

Winding Numbers, Complex Currents, and Non-Hermitian Localization

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The nature of extended states in disordered tight binding models with a constant imaginary vector potential is explored. Such models, relevant to vortex physics in superconductors and to population biology, exhibit a delocalization transition and a band of extended states even for a one dimensional ring. Using an analysis of eigenvalue trajectories in the complex plane, we demonstrate that each delocalized state is characterized by an (integer) winding number, and evaluate the associated complex current. Winding numbers in higher dimensions are also discussed.

There is a growing interest in the spectra of random non-Hermitian matrices [1]. Considerable attention has focused on a particularly simple class of tight binding Anderson models with a constant imaginary vector potential, inspired by the physics of vortex matter [2]. These models exhibit a sharp delocalization transition even in one and two dimensions. Similar operators, represented by large real, asymmetric sparse matrices, arise in theories of population biology in random media with convection [3], and in many other contexts [4]. Delocalized eigenmodes arise in response to a sufficiently large asymmetry parameter, accompanied by eigenvalues which escape in conjugate pairs from the real axis into the complex plane [2].

Although the initial work relied heavily on numerical investigations [2], there has been considerable recent analytic progress, particularly in one dimension. Brouwer et. al. calculate explicitly finite size effects and the bubble of complex eigenvalues representing extended states in the center of the band in the limit of weak disorder [5]. Brezin and Zee [6] have obtained exact results for the density of states for the special case of Lorentzian site randomness embodied in the Lloyd model [10]. Goldsheid and Khoruzhenko discuss this model and show more generally that the eigenvalues are distributed along curves in the complex plane, providing analytic formulas relating the spectrum to the properties of a reference Hamiltonian with no asymmetry [7]. The nontrivial behavior which results for *large* asymmetry parameters in two and three dimensions has been studied analytically via renormalization group calculations and a mapping onto Burgers equation [3].

In this paper we study the eigenfunctions and complex currents associated with the band of extended states in one dimension. Unlike delocalized states in Hermitian disordered systems (where the eigenfunctions can always be chosen to be real), we show that these complex eigenfunctions are characterized by a conserved winding number n even when the disorder is strong. Such topological quantum numbers can be used to label the eigenvalue spectrum $\epsilon_n(g)$, where g is the asymmetry parameter, i.e., the imaginary vector potential. A study of the eigenvalue trajectories as a function of g then leads to complex

currents $J_n = -i \frac{\partial \epsilon_n}{\partial g}$, which determine the average tilt dr/dz of the vortex trajectory $r(z)$ in a superconductor with columnar pins in the presence of a perpendicular magnetic field proportional to g [2]. Assuming for simplicity a periodic box of size L_z in the column direction, one has, for example,

$$\langle \frac{dr(z)}{dz} \rangle = -i \frac{\sum_n J_n e^{-\epsilon_n L_z / T}}{\sum_n e^{-\epsilon_n L_z / T}}, \quad (1)$$

where the brackets denote a thermal average at temperature T . We shall also argue that winding numbers can be used to label extended states for *large* asymmetry parameters in two or more dimensions.

The one dimensional non-Hermitian tight binding model “Hamiltonian” in a basis of N sites localized at positions $\{r_j\}$ reads,

$$\mathcal{H}_{i,j} = -\frac{1}{2} w_j (e^{h_j} \delta_{i+1,j} + e^{-h_j} \delta_{i-1,j}) - V_j \delta_{ij}, \quad (2)$$

where we assume periodic boundary conditions and the $\{h_j\}$ are real asymmetry parameters. When exponentiated, this operator describes the transfer matrix for a flux line with columnar defects in a cylindrical shell [2]. The Liouville operator describing population growth on a lattice in an inhomogeneous environment with convection is given by $\mathcal{L} = -\mathcal{H}$ [3]. The zero mean random potential V_j is chosen independently for each site, and arises from the variations of columnar pin diameters (vortex matter) or inhomogeneous growth rates (population biology). As discussed in [2], the randomness in the off-diagonal hopping is due to the irregular spacing between columnar defects in superconductors. If a_j is the spacing between nearest neighbor columns j and $j+1$, then the $w_j = w_0 \exp(-\gamma a_j)$ and $h_j = g a_j / a$, where γ and $w_0 > 0$ are constants, a is the average lattice constant, and g is proportional to the field component which tries to tip vortices away from the columns. In models of population growth, w_j describes diffusion between inhomogeneously distributed population centers and h_j represents a fluctuating convection velocity with average value g and second moment σ . We impose periodic boundary conditions in

the space-like direction i.e., the $N + 1$ -st site is identified with the first site of the sample.

Upon carrying out a similarity transformation on the Liouville operator, $\mathcal{L}' = S\mathcal{L}S^{-1}$, where

$$S_{i,j} = \delta_{i,j} \exp\left[\sum_{k=1}^{j-1} h_k - \frac{(j-1)}{N} \sum_{k=1}^N h_k\right], \quad (3)$$

one finds that the spectrum will be the same as that of a Liouvillian with a *uniform* asymmetry parameter, namely $\mathcal{L}' = -\mathcal{H}'$ with

$$\mathcal{L}'(g) = \begin{pmatrix} V_1 & \frac{w_1}{2}e^g & 0 & \dots & \frac{w_N}{2}e^{-g} \\ \frac{w_1}{2}e^{-g} & V_2 & \frac{w_2}{2}e^g & \dots & 0 \\ 0 & \frac{w_2}{2}e^{-g} & V_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{w_N}{2}e^g & 0 & 0 & \dots & V_N \end{pmatrix}. \quad (4)$$

where $g = \frac{1}{N} \sum_{j=1}^N h_j$. Eqs. (3 - 4) generalizes the gauge transformation used to describe *localized* states in, e.g., Ref. [2] and [3]. The sample to sample fluctuations of g about its mean fall off like σ/\sqrt{N} [8]. As N grows one may thus replace the fluctuating quantity h_i by a disorder independent average value g , as we shall do in the rest of this paper. This result implies *universality* in the response to random convection in 1D growth models - nonuniform convection velocities may be mapped into a uniform average velocity via the transformation (3), provided one applies this similarity transformation to the eigenfunctions as well.

For a 1D ring with random parameters $\{V_i\}, \{w_i\}$ and $g = 0$, all the eigenfunctions of (4) are real and localized, and its eigenvalues are real and discrete [9]. We assume a large but finite chain, such that although the spectrum is discrete, the length of the chain is much larger than the maximal localization length of an eigenmode.

In a typical symmetric one-dimensional disordered system with $g = 0$, the localization length ξ is largest at the center of the band and smaller at the tails. The criterion for delocalization of the asymmetric system is $\kappa a < g$, where $\kappa \equiv 1/\xi$ [13]. As a result, pairs of complex energies representing delocalized states first appear as a “bubble” at the center of the band, which then spreads into the band tails. To study the complex currents associated with delocalized states we follow Refs. [5-7] and exploit the relation between the complex spectrum of the asymmetric problem with $g \neq 0$ and the real eigenvalues of a “background” localized problem with $g = 0$, and same realization of disorder.

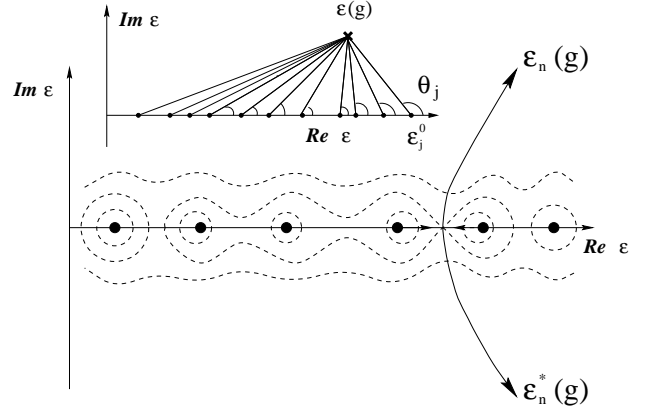


FIG. 1. Eigenvalue trajectories in the complex plane. Inset: Angles entering Eq. (6) for a given $\epsilon(g)$.

The condition for a complex number $\epsilon = \epsilon_R + i\epsilon_I$ to be an eigenvalue of the matrix $\mathcal{L}'(g)$ is $\text{Det}[\epsilon - \mathcal{L}'(g)] = 0$. Eq. (4) implies that ϵ is an eigenvalue only if [5,12]

$$\begin{aligned} \text{Det}[\epsilon - \mathcal{L}'(g=0)] &= \prod_{i=1}^N (\epsilon - \epsilon_i^0) \\ &= 2[\cosh(gN) - 1] \prod_{j=1}^N \left(\frac{-w_j}{2}\right) \end{aligned} \quad (5)$$

where the $\{\epsilon_i^0\}$ are the (real) eigenvalues of the background matrix $\mathcal{L}'(g=0)$. To extract winding numbers, we first observe that the right hand side of (5) is real and positive for even N , while for odd N it is real and negative. As a result, each complex ϵ should satisfy (see inset to Fig. 1)

$$\sum_i \cot^{-1}\left(\frac{\epsilon_R - \epsilon_i^0}{\epsilon_I}\right) = p\pi, \quad (6)$$

with $p = 2n$ for even N , $p = 2n + 1$ for odd N , where n is an integer, and the function $\cot^{-1}(x)$ varies from π to 0 as x goes between $-\infty$ and ∞ .

As $\epsilon_I \rightarrow 0$, each term in Eq. (6) gives π for every eigenvalue ϵ_i^0 to the right of ϵ_R , and zero for each eigenvalue to the left. To satisfy (6) for even (odd) N and a given n , the eigenvalue must leave the real axis and enter the complex plane at the gap between the $2n$ -th and the $2n+1$ -th ($2n+1$ -th and the $2n$ -th) eigenvalues of the $g = 0$ “background” system. We call n the *index* of the trajectory $\epsilon_R(g) + i\epsilon_I(g)$ in the complex plane.

For odd N , we see immediately from (6) that the rightmost eigenvalue (with $p = n = 0$) must remain real, consistent with Perron-Frobenius theorem [11]. The corresponding nodeless eigenfunction corresponds to the ground state of $\mathcal{H}' = -\mathcal{L}'$. For N even, particle-hole symmetry implies that both the rightmost and leftmost eigenvalues are always real. More generally, for a fixed value of n , the set of all $[\epsilon_R(g), \epsilon_I(g)]$ satisfying Eq. (6) defines a curve in the complex plane, as illustrated in Fig. 1. Henceforth, we assume N even for simplicity.

A more complete description of the eigenvalue trajectory results from taking the logarithm of the modulus of Eq. (5). In the limit $Ng \gg 1$ one finds a second constraint on $\epsilon(g)$, namely, [5-7]

$$|g| = -\ln\left(\frac{w}{2}\right) + \frac{1}{N} \sum_i \ln(|\epsilon - \epsilon_i^0|) \quad (7)$$

where $|\epsilon - \epsilon_i^0| = \sqrt{(\epsilon_R - \epsilon_i^0)^2 + \epsilon_I^2}$ and $w = 2[\prod_{j=1}^N \frac{w_j}{2}]^{\frac{1}{N}}$.

This constraint is described graphically in Fig. 1, which shows schematically the level curves defined by (7) near the band center for three values of g . These are lines of constant potential for an equivalent 2d electrostatic problem with charges at the positions of the localized point spectrum for $g = 0$. When g is small, the constraint is solved by eigenvalue pairs on the real axis in the gaps between neighboring ϵ_j^0 's indexed by $p = 2n$. The departure of the eigenvalues from their $g = 0$ values is initially exponentially small in the system size [2]. As g increases, however, successive pairs of eigenvalues eventually merge at a saddle point in the potential contours and detach from the real axis at right angle. For a given real energy ϵ_R , Thouless has defined an energy dependent inverse localization length $\kappa(\epsilon_R)$ for the associated Hermitian problem [12],

$$\kappa(\epsilon_R) = -\ln\left(\frac{w}{2}\right) + \frac{1}{N} \sum_i \ln(|\epsilon_R - \epsilon_i^0|). \quad (8)$$

Upon comparing with Eq. (7), we see that ϵ_I becomes nonzero whenever $|g| > \kappa(\epsilon_R')$ where $(\epsilon_R', 0)$ is the detachment point of the eigenvalue pair.

As g increases above $\kappa(\epsilon_R')$, Eqs. (6) and (7) define a unique pair of complex eigenvalue trajectories $\epsilon_n(g)$ and $\epsilon_n^*(g)$ for every value of n . Upon passing to the limit $N \rightarrow \infty$, the spectrum $\{\epsilon_j^0\}$ for $g = 0$ closes up, and is described by a density of states $\rho_0(\lambda)$. Eq. (6) and (7) may then be recombined into a single complex equation, namely

$$\int_{-\infty}^{\infty} d\lambda \rho_0(\lambda) \ln[\epsilon_n(g) - \lambda] = \ln\left(\frac{w}{2} e^{|g|}\right) + i\pi \left(\frac{2n}{N}\right). \quad (9)$$

where $\ln[\epsilon(g) - \lambda] = \ln[|\epsilon_R + i\epsilon_I - \lambda|] + i \cot^{-1}[(\epsilon_R - \lambda)/\epsilon_I]$. In the limit $N \gg 1$ and for a density of states function symmetric around the special detachment point with $\epsilon_R' = 0$, there is a purely imaginary trajectory of the form $\epsilon(g) = i\epsilon_I(g)$ with $n = N/2$. For fixed g , Eq. (9) thus leads to an implicit formula for the “height” ϵ_I^{max} of the bubble of complex eigenvalues in the center of the band, namely, $\frac{1}{2} \int d\lambda \rho(\lambda) \ln[\lambda^2 + (\epsilon_I^{max})^2] = |g| + \ln(\frac{w}{2})$. This integral vanished, as expected [6], in the “one way” limit, $g \rightarrow \infty$ with $\frac{w}{2} e^{|g|} = 1$. For other detachment points, the eigenvalue trajectories curve to the left or right as required by the constraint (6) (See Fig. 1).

The analysis above suggests that the imaginary parts of most $\{\epsilon_n(g)\}$ diverge as $|g| \rightarrow \infty$. It is then expected

that *all* eigenfunctions $\phi_n(j)$ are approximately plane waves, $\phi_n(j) \sim \exp(ik_n r_j/a)$, with free particle eigenvalue energies $\epsilon_n(g) = w \cos(k_n + ig)$ [2,3]. For large $|\epsilon_I(g)|$, Eq. (9) leads to

$$\epsilon_n(g) \approx \frac{w}{2} \exp(|g| + 2i\pi n/N). \quad (10)$$

Comparison with the free particle spectrum at large $|g|$ shows immediately that the index n of the eigenvalue trajectory and the wavevector are related, $k_n = 2\pi n/N$. These wave eigenfunctions spiral around the origin in the complex plane as one moves along the 1d lattice of the tight binding model sites, leading to a well defined winding number n . The winding number associated with the eigenvalue $\epsilon_n^*(g)$ in the lower half plane is then $-n$.

As g decreases, the associated delocalized wave function $\phi_n(j; g)$ must *remain* nonzero at every site: If $\phi_n(j; g)$ were in fact exactly zero at some site i , it can be shown that the state is localized with a real eigenvalue and eigenfunction by mapping all effects of the asymmetry onto the special site i via a transformation like (3). Thus, the winding number must be *preserved* as g decreases, i.e., the winding number is a topological invariant along an eigenvalue trajectory. The projection of such a delocalized eigenfunction is illustrated in the inset to Fig. 2.

As $N \rightarrow \infty$, we can replace the winding index n by the continuous variable $k_n = \frac{2\pi}{N}n$. Eq. (9) then shows quite generally that the complex spectrum is a function only of the combination $g + ik$, i.e., $\epsilon_n(g) = \epsilon(g + ik)$. It then follows from the Cauchy-Riemann relations that all eigenvalue trajectories are at right angles to the lines of constant g , $\frac{d\epsilon(g)}{dg} = -i \frac{d\epsilon(g)}{dk}$.

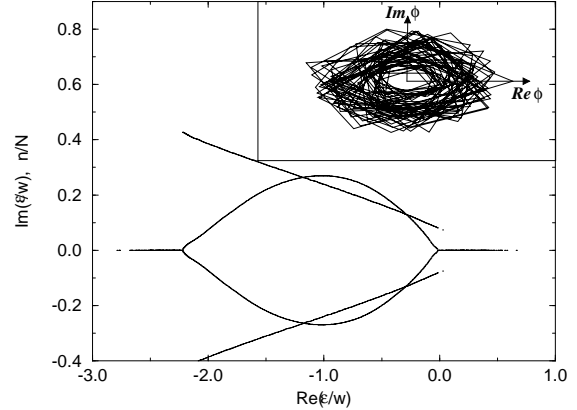


FIG. 2. Spectrum and winding numbers for $N = 1000$ with V_j uniformly distributed in the interval $[-1, 1]$, and $g = 0.4$. Inset: Projection onto the complex plane of eigenfunction with $N = 200$, $g = 1$, and $n = 100$.

An explicit formula for the complex current results, moreover, from differentiating Eq. (9) with respect to g ,

$$J[\epsilon_n(g)] = -i \frac{\partial \epsilon_n(g)}{\partial g} = \left[i \int d\lambda \frac{\rho_0(\lambda)}{\epsilon_n(g) - \lambda} \right]^{-1} \equiv -i G_0^{-1}[\epsilon_n(g)] \quad (11)$$

Evidently, the current associated with a particular complex eigenvalue $\epsilon_n(g)$ is determined by the Green's function of the $g = 0$ problem. As an application of our results, we note that the disorder averaged Green's function for the Lloyd model, with Lorentzian site disorder, $Pr(V_j) = \frac{1}{\pi} \frac{\gamma}{(V_j)^2 + \gamma^2}$ is $[6,10] < G_0(E) > = [(E + i\gamma) - t]^{-1/2}$, and complex energy spectrum for extended states with $\text{Im} \epsilon_n > 0$, $\epsilon_n(g) = w \cos(k_n + ig) - i\gamma$. It then follows immediately from Eq. (11) that the disordered averaged inverse complex current $J_n = | < J_n^{-1} >^{-1} |$ is

$$J_n = -i w \sin(k_n + ig) \quad (12)$$

in agreement with a direct calculation of the current from the spectrum itself. If we focus attention on $\text{Im} J$ as a function of $\text{Re} \epsilon$, we obtain $\text{Im} J = -w \tanh(g) \text{Re} \epsilon$, in excellent qualitative agreement with the results obtained for the imaginary current with bounded disorder in Ref. [2] for different values of g . Although we have checked our results with the Lloyd model, we stress that Eq. (11) expressing the complex current in terms of the Green's function of the Hermitian reference system is much more general.

Fig. (2) shows a typical eigenvalue spectrum for \mathcal{L}' in one dimension superimposed on the winding numbers associated with the extended eigenfunctions. The winding numbers are exactly $\pm N/2$ at the band center, and their magnitudes decrease monotonically as one moves toward the upper edge of the band, i.e., toward the lowest energies of the corresponding Hamiltonian. The winding numbers remain finite up to the mobility edge separating complex and real eigenvalues, but become undefined in the band tails, where all wavefunctions are real. In this case, the winding index classification can be replaced by counting the nodes in the localized wavefunction [12]. The size of the inner excluded region of the projected wavefunction shown in the inset shrinks to zero at the mobility edge. The winding number at the mobility edge decreases smoothly to zero as g increases and all states become delocalized.

Given that a well defined topological quantum number (which plays the role like that of the momentum) exists for the extended states in the one dimensional non-Hermitian localization problem, it is interesting to speculate whether similar topological invariants exist for delocalized states in higher dimensions. For large g in $D \geq 2$, disorder produces nontrivial changes in the spectrum, but the eigenfunctions are still slightly perturbed plane waves [3]. For $N \times N$ lattices with toroidal boundary conditions, it seems likely that the winding number (n_x, n_y) embodied in the wavevector $\mathbf{k} = (k_x, k_y) = (2\pi n_x/N, 2\pi n_y/N)$

is initially preserved as g decreases from large values. However, no detailed analysis exists of how these extended states in $D = 2$ become localized as their eigenvalues approach the real axis, and we cannot be certain if the winding number remains strictly invariant along a trajectory. In this context, it is interesting to note that a recent analysis of delocalized wavefunctions for the non-Hermitian problem in $D = 2$ predicts algebraic decay of the wavefunction amplitude [14], similar to the quasi-long range order below the vortex unbinding transition in statistical mechanics. Vortex pairs may also be involved in changes of the vector winding number of delocalized wave functions for $d = 2$, similar to the decay of supercurrents in Helium films at finite temperature [15].

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